Regularity and entropy of non-commutative random variables

Ian Charlesworth

April 1, 2022

The Setting

$(\mathcal{A}, \tau)$ a non-commutative probability space

$\mathcal{A}$ a von Neumann algebra:

$\mathcal{A} \subseteq B(\mathcal{H})$, $\mathcal{A} = \mathcal{A}^*$, $1 \in \mathcal{A}$

$\mathcal{A}$ SOT-closed ($\Leftrightarrow \mathcal{A} = \mathcal{A}''$, its double commutant)

If $Z(\mathcal{A}) = \mathbb{C}1$, $\mathcal{A}$ is called a factor

$\tau : \mathcal{A} \to \mathbb{C}$ a faithful trace

τ linear

$\tau(1) = 1$

$\tau(x^*x) \geq 0$, with identity only when $x = 0$

$\tau(xy) = \tau(yx)$

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Examples:

- $\left(L^\infty(\Omega, d\mathbb{P}), \mathbb{E}\right)$ where $L^\infty(\Omega, d\mathbb{P}) \subseteq B\left(L^2(\Omega, d\mathbb{P})\right)$
- $\left(M_n(\mathbb{C}), tr\right)$ where $M_n(\mathbb{C}) \cong B(\mathbb{C}^n)$
- $\left(M_n\left(L^\infty(\Omega, d\mathbb{P})\right), \mathbb{E}\circ tr\right)$
- $\left(L(\Gamma), \tau\right)$ where $\Gamma \overset{\lambda}{\curvearrowright} \ell^2(\Gamma)$ by left translation,

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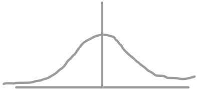
The Dictionary Slide

Mandatory

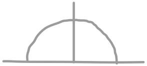
$$(L^\infty(\Omega, d\mathbb{P}), \mathbb{E}) \rightsquigarrow (\mathcal{A}, \tau)$$

$$(\mathbb{C}^n, \text{unif}) \rightsquigarrow (M_n(\mathbb{C}), \mathrm{tr})$$

Independence, $\otimes$ $\rightsquigarrow$ Free independence, $*$



Gaussian distribution $\rightsquigarrow$ Wigner Semicircle distribution



$$\mu_X \in \mathrm{Prob}(\mathbb{R}^{\# X}) \rightsquigarrow \begin{aligned} \mu_X : \mathbb{C}\langle T \rangle &\longrightarrow \mathbb{C} \\ p &\longmapsto \tau(p(X)) \end{aligned}$$

Shannon entropy $\rightsquigarrow$ Free entropy

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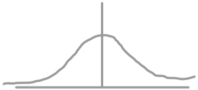

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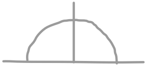

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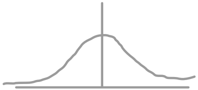

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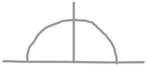

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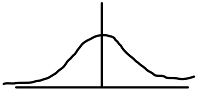

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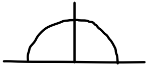

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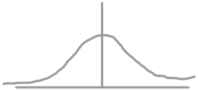

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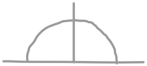

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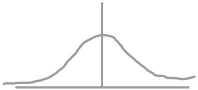

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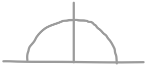

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Suppose $(M_i^{(N)})_i$ are independent random $N \times N$ matrices which are self-adjoint and unitarily invariant, so that

$$\mu_{M^{(N)}} \xrightarrow[N \to \infty]{} \mu_{\mathbf{X}} \quad \text{for some tuple } \mathbf{X}.$$

Then $X_1, \dots, X_n$ are freely independent.

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Entropy

"You should call it entropy, for two reasons. In the first place your uncertainty function has been used in statistical mechanics under that name, so it already has a name. In the second place, and more important, no one knows what entropy really is, so in a debate you will always have the advantage."

—John von Neumann, to Claude Shannon

(Quoted from the retelling in [Tribus McIrvine 1971].)

Entropy

$S = k \log W$ on Boltzmann's grave

Microstates free entropy

Idea: $\mu_{\mathbf{X}}$ is a macrostate, matrices correspond to microstates

$$\Gamma(\mathbf{X}; k, \mathcal{U}) = \left\{ \mathbf{M} \in \left(M_k^{s.a.}(\mathbb{C})\right)^{\#\mathbf{X}} \,\middle|\, \mu_{\mathbf{M}} \in \mathcal{U} \right\}$$

$\hookrightarrow$ weak open neighbourhood of $\mu_{\mathbf{X}}$

$$\chi(\mathbf{X}) = \inf_{\mathcal{U}} \limsup_{k \to \infty} \left[\frac{1}{k^2} \log \lambda\left(\Gamma(\mathbf{X}; k, \mathcal{U})\right) + \frac{\#\mathbf{X} \log k}{2} \right]$$

$$\chi(X) = \iint \log|s-t| \, d\mu_X(s) \, d\mu_X(t) + C$$

For nice enough $\mathbf{F}$, $\chi(\mathbf{F}(\mathbf{X})) = \chi(\mathbf{X}) + \#\mathbf{X} \log\left(|\det| \mathcal{J}\mathbf{F}(\mathbf{X})\right)$

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Entropy can detect properties of von Neumann algebras

Suppose $(\mathcal{A}, \tau)$ is a factor. $\mathcal{A}$ has Property Γ if there is a bounded sequence $t_i \in \mathcal{A}$ so that for all $s \in \mathcal{A}$, $\tau([t_i, s]^*[t_i, s]) \to 0$ but $\tau((t_i - \tau(t_i))^*(t_i - \tau(t_i))) \not\to 0$.

If $W^*(\mathbf{X})$ has Property Γ, (and $\#\mathbf{X} > 1$) then $\chi(\mathbf{X}) = -\infty$.

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Classically, $\frac{d}{dt} S(\mathbf{X} + \sqrt{t}\,\mathbf{N})\Big|_{t=0} = \mathcal{I}(\mathbf{X}) := \| J^* 1 \|^2_{L^2(\Omega, d\mathbb{P})^n}$

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Free entropy dimension

$$\delta^*(X) = \#X - \liminf_{\varepsilon \to 0} \frac{\chi^*(X + \sqrt{\varepsilon}\, S)}{\log \sqrt{\varepsilon}}$$

$\delta_0(X)$ – asymptotic packing number for $\Gamma(X; k, u)$

[Bercovici Voiculescu] X, Y free.

$$\mu_{X+Y}(\{t\}) = \alpha > 0 \iff \exists\, s_1, s_2 \quad t = s_1 + s_2, \quad \mu_X(\{s_1\}) + \mu_Y(\{s_2\}) - 1 = \alpha$$

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[C Shlyakhtenko; Mai Speicher Weber]

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$\delta^*(\mathbf{X}) = \#\mathbf{X} \Rightarrow ev_{\mathbf{X}} : \mathbb{C}\langle T \rangle \longrightarrow W^*(\mathbf{X})$ extends to the free skew field $\mathbb{C}\langle\!\langle T \rangle\!\rangle$

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Open problems:

- Say p is a (non-constant) polynomial and $\chi^*(\mathbf{X}) > -\infty$. Is $\chi(p(\mathbf{X})) > -\infty$?
- Is δ^* a von Neumann algebra invariant? Is δ_0?