Non-commutative Information Theory
An Overview

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Background

Motivation:

Understand von Neumann algebras.

$\hookrightarrow$ $\mathcal{M} \subseteq B(\mathcal{H})$ unital $*$-closed subalgebra

with: (i) $\mathcal{M} = \overline{\mathcal{M}}^{SOT}$

(ii) $\mathcal{M} = \overline{\mathcal{M}}^{WOT}$

(iii) $\mathcal{M} = (\mathcal{M}')'$ the bicommutant

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Examples

- $M_n(\mathbb{C})$
- $\mathcal{B}(\mathcal{H})$
- $\mathrm{Diag}(M_n(\mathbb{C})) \cong \mathbb{C}^n$
- Block diagonal subalgebra of $\mathcal{B}(\mathcal{H})$
  (If (p_i) are orthogonal projections with $\sum_i p_i = 1$,
  $$\bigoplus_i p_i \mathcal{B}(\mathcal{H}) p_i \quad (\ldots))$$
- Γ countable, discrete group
  $$\lambda : \mathbb{C}[\Gamma] \hookrightarrow \mathcal{B}(\ell^2(\Gamma))$$
  $$L\Gamma := \overline{\lambda(\mathbb{C}[\Gamma])}^{SOT}$$
- $L^\infty(\Omega, \mathbb{P}) \leq L^2(\Omega, \mathbb{P})$

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- $L(\mathbb{Z}) \cong L^\infty(\mathbb{T}) \cong L^\infty(\mathbb{T}^2) \cong L(\mathbb{Z}^2)$
- Any two separable amenable discrete ICC groups generate isomorphic vNas
  $(S_\infty, (\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}, \ldots)$
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Non-commutative probability theory

If commutative vNa's are $L^\infty(\Omega, \mathbb{P})$,
view arbitrary vNa's as spaces of
"non-commutative random variables".

A tracial state on $\mathcal{M}$ is a linear map
$\tau : \mathcal{M} \longrightarrow \mathbb{C}$ which is

positive $(\tau(a^*a) \geq 0)$
unital $(\tau(1) = 1)$
tracial $(\tau(ab) = \tau(ba))$

We'll generally also assume

faithful $(\tau(a^*a) = 0 \Rightarrow a = 0)$
normal $(\tau(\sup x_i) = \sup \tau(x_i)$ for (x_i) bounded increasing positive net$)$

Examples

- $\mathrm{tr} : M_n(\mathbb{C}) \to \mathbb{C}$
- $\langle \delta_e, \cdot\, \delta_e \rangle : L\Gamma \longrightarrow \mathbb{C}$
- $\mathbb{E} : L^\infty(\Omega, \mathbb{P}) \longrightarrow \mathbb{C}$

If $x = x^* \in (M, \tau)$, x generates a commutative vNa.
What is the corresponding probability space?

$L^\infty(\sigma(x), \tau \circ P_x)$ where P_x is the spectral measure:

$$f(x) = \int_{\sigma(x)} f(\lambda)\, dP_x(\lambda)$$

Free Independence

$(\mathcal{A}_i)_{i \in I}$ $*$-subalgebras of $(\mathcal{M}, \tau)$ are freely independent if whenever

$$i_1 \neq i_2 \neq \cdots \neq i_n \quad \text{in } I$$

$$x_k \in \mathcal{A}_{i_k} \cap \ker(\tau)$$

we have $x_1 \cdots x_k \in \ker(\tau)$

Specifies all mixed moments by centring and recursion:

$$0 = \tau\big((x_1 - \tau(x_1)1) \cdots (x_n - \tau(x_n)1)\big)$$

$$= \tau(x_1 \cdots x_n) - \tau(x_1)\tau(1\, x_2 \cdots x_n) - \cdots$$

Example

$$0 = \tau\Big((x-\tau(x)1)(y-\tau(y)1)\Big)$$

$$= \tau(xy) - \tau(x)\tau(1y) - \tau(x1)\tau(y) + \tau(x)\tau(y)\tau(1)$$

$$\Rightarrow \tau(xy) = \tau(x)\tau(y)$$

More calculations

$$\Rightarrow \tau(xyxy) = \tau(x^2)\tau(y)^2 + \tau(x)^2\tau(y^2) - \tau(x)^2\tau(y)^2$$

In particular,

$$\tau\big((xy-yx)^*(xy-yx)\big) = 2\big(\tau(x^2)-\tau(x)^2\big)\big(\tau(y^2)-\tau(y)^2\big)$$

$$= 2\,\mathrm{Var}(x)\,\mathrm{Var}(y)$$

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Example

`Click me!`

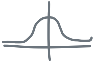
The Mandatory Dictionary Slide

$\left(L^\infty(\Omega, \mathbb{P}), \mathbb{E}\right)$ —— $(\mathcal{M}, \tau)$

$\left([n], \frac{1}{n}\sum \delta_i\right)$ —— $\left(M_n(\mathbb{C}), \mathrm{tr}\right)$

Independence / $\otimes$ —— Free independence / $*$

$(x_i)_i$ i.i.d. —— $(x_i)_i$ f.i.i.d.

$\frac{1}{\sqrt{N}}\sum x_i \to$  —— $\frac{1}{\sqrt{N}}\sum x_i \to$ 

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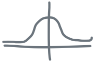
$(x_i)_i$ i.i.d. $\frac{1}{\sqrt{N}}\sum x_i \to$ 	$(x_i)_i$ f.i.i.d. $\frac{1}{\sqrt{N}}\sum x_i \longrightarrow$ 
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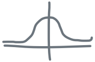
$(x_i)_i$ i.i.d. $\frac{1}{\sqrt{N}}\sum x_i \to$  — $(x_i)_i$ f.i.i.d. $\frac{1}{\sqrt{N}}\sum x_i \to$ 

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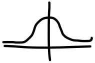
$(x_i)_i$ i.i.d. $\frac{1}{\sqrt{N}}\sum x_i \to$ 	$(x_i)_i$ f.i.i.d. $\frac{1}{\sqrt{N}}\sum x_i \to$ 
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Characterize $W^*(x_1, \ldots, x_n)$ based on the mixed moments of $x_1, \ldots, x_n$.

(E.g. $W^*(x_1, \ldots, x_n)$ abelian $\Leftrightarrow \tau\left(\sum_{i,j} |[x_i, x_j]| \right) = 0$)

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Microstates:

$$\chi(x_1, \dots, x_n) = \sup_{R>0} \inf_{\mathcal{U} \ni \mu_x} \limsup_{k \to \infty} \left(\frac{n \log k}{2} + \frac{1}{k^2} \log \lambda \left\{ X \in M_k^{sa}(\mathbb{C})^n \,\middle|\, \begin{matrix} \|X\| < R \\ \mu_X \in \mathcal{U} \end{matrix} \right\} \right)$$

= asymptotic log volume of matricial approximations

If $U \sim \mathrm{Haar}(\mathcal{U}(N))$, $X \in M_N(\mathbb{C})$ deterministic,

X and UXU^* are almost free

If $W^*(x_1, \dots, x_n)$ has a central projection,

matricial approximations are almost simultaneously block diagonalizable.

[Voiculescu, many others]

Free Information Theory

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de Bruijn's formula, oversimplified:

if $X(t) = X + \sqrt{t}\,N$,

$$\frac{d\,S(X(t))}{dt} = \frac{1}{2}\left\| \left(\frac{d}{dt}\right)^{*} 1 \right\|^{2}_{L^2(\mathbb{R},\, \mu_{X(t)})}$$

Consequence:

to understand entropy, we can look at the derivative as an unbounded operator on L^2.

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[C, Nelson]

$$A = \mathbb{C}\langle x_1, \dots, x_n \rangle \subseteq W^*(x_1, \dots, x_n) = \mathcal{M}$$

$$\mathrm{Der}_{1\otimes 1}(A) = \left\{ \delta : A \to L^2(\mathcal{M},\tau) \otimes L^2(\mathcal{M},\tau) \;\middle|\; \begin{array}{l} \delta(xy) = x\delta(y) + \delta(x)y \\ 1\otimes 1 \in \mathrm{dom}(\delta^*) \end{array} \right\}$$

$$= \left\{ \text{"nice" derivations } A \to L^2(\mathcal{M},\tau) \otimes L^2(\mathcal{M},\tau) \right\}$$

can be equipped with the structure of an $\mathcal{M}$-$\mathcal{M}$ bimodule, and has a corresponding dimension in $[0, n]$, denoted $\sigma(A)$.

The bimodule structure depends on $x_1, \dots, x_n$, but $\sigma(A)$ does not!

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Free Stein Dimension

Can bootstrap many results: e.g.,

$\sigma(x_1, \ldots, x_n) = n \Rightarrow$ no n.c. rational relations

(heavily using work of [Connes-Shlyakhtenko]

and [Mai-Speicher-Yin])