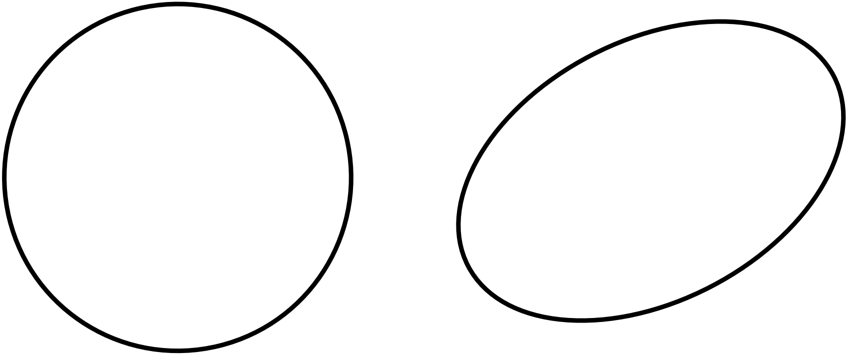
Mixtures of classical and free independence

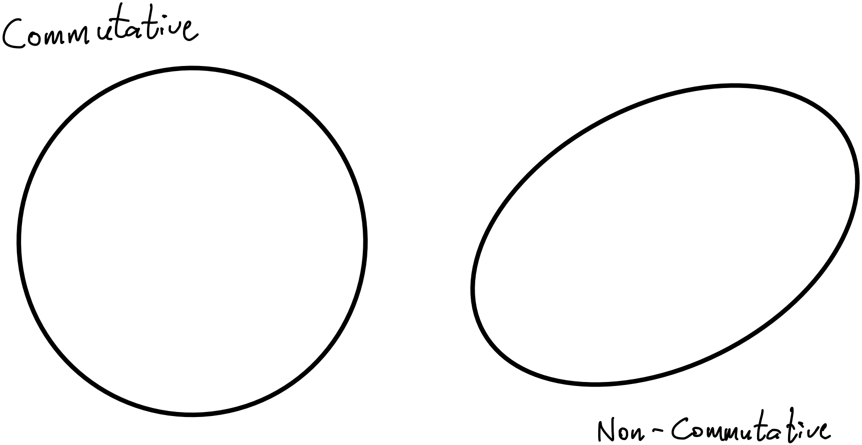
Ian Charlesworth

Prifysgol Caerdydd / Cardiff University

2026-06-25

Overview




Commutative
Non-Commutative

Overview

Commutative

$\Gamma_1 \times \Gamma_2$

$\mathbb{F}[x_1, x_2, \ldots]$

$L^\infty(\Omega_1, \mathbb{P}_1) \otimes L^\infty(\Omega_2, \mathbb{P}_2)$

$\mathbb{E}_1 \otimes \mathbb{E}_2$

$\Gamma_1 * \Gamma_2$

$\mathbb{F}\langle X_1, X_2, \ldots \rangle$

$(\mathcal{M}_1, \varphi_1) * (\mathcal{M}_2, \varphi_2)$

$\varphi_1 * \varphi_2$

Non-Commutative

Overview

Commutative

$\Gamma_1 \times \Gamma_2$

$\mathbb{F}[x_1, x_2, \dots]$

$L^\infty(\Omega_1, \mathbb{P}_1) \otimes L^\infty(\Omega_2, \mathbb{P}_2)$

$\mathbb{E}_1 \otimes \mathbb{E}_2$

$\xleftrightarrow{\ ?\ }$

$\Gamma_1 * \Gamma_2$

$\mathbb{F}\langle X_1, X_2, \dots \rangle$

$(\mathcal{M}_1, \varphi_1) * (\mathcal{M}_2, \varphi_2)$

$\varphi_1 * \varphi_2$

Non-Commutative

Combinatorics of words

$A = \{1, 2, \ldots, n\} = [n]$ finite set

A^* set of words (finite sequences) with letters in A

$A^{\otimes}$ set of commutative words (monomials) with letters in A

E.g. $\varepsilon, 1, 1^2 2, 121, 21^2, \ldots \in A^*$

$$A^{\otimes} = A^* / \{ij \sim ji\}$$

$$\sum_{w \in A^*} X_w = (1 - X_1 - X_2 - \cdots - X_n)^{-1} \in \mathbb{Z}\langle\langle X_1, \ldots, X_n \rangle\rangle$$

$$\sum_{[w] \in A^{\otimes}} X_w = (1 - X_1)^{-1}(1 - X_2)^{-1} \cdots (1 - X_n)^{-1} \in \mathbb{Z}[[X_1, \ldots, X_n]]$$

Combinatorics of words

$A = \{1, 2, \ldots, n\} = [n]$ finite set

A^* set of words (finite sequences) with letters in A

$A^{\otimes}$ set of commutative words (monomials) with letters in A

E.g. $\varepsilon, 1, 1^2 2, 121, 21^2, \ldots \in A^*$

$$A^{\otimes} = A^* / \{ij \sim ji\}$$

$$\sum_{w \in A^*} X_w = (1 - X_1 - X_2 - \cdots - X_n)^{-1} \in \mathbb{Z}\langle\langle X_1, \ldots, X_n \rangle\rangle$$

$$\sum_{[w] \in A^{\otimes}} X_w = (1 - X_1)^{-1}(1 - X_2)^{-1} \cdots (1 - X_n)^{-1} \in \mathbb{Z}[[X_1, \ldots, X_n]]$$

Combinatorics of words

$A = \{1, 2, \ldots, n\} = [n]$ finite set

A^* set of words (finite sequences) with letters in A

$A^{\otimes}$ set of commutative words (monomials) with letters in A

E.g. $\varepsilon, 1, 1^2 2, 121, 21^2, \ldots \in A^*$

$$A^{\otimes} = A^* / \{ij \sim ji\}$$

$$\sum_{w \in A^*} X_w = (1 - X_1 - X_2 - \cdots - X_n)^{-1} \in \mathbb{Z}\langle\langle X_1, \ldots, X_n \rangle\rangle$$

$$\sum_{[w] \in A^{\otimes}} X_w = (1 - X_1)^{-1}(1 - X_2)^{-1} \cdots (1 - X_n)^{-1} \in \mathbb{Z}[[X_1, \ldots, X_n]]$$

Combinatorics of words

[Cartier — Foata 1969, Paraphrased]

Let $G = (A, E)$ be a (simple) graph

$$A^G = A^* / \{ uv \sim vu \mid (u,v) \in E \}$$

$12321 \sim 2^2131 \sim 132^21$

$1234 \sim 1324$

$$\sum_{[w] \in A^G} X_w = \Big(\sum_{\substack{K \subseteq A \\ \text{clique}}} \prod_{v \in K} (-X_v) \Big)^{-1} \in \mathbb{Z}[[X_1, \ldots, X_n]]$$

or

$$\mathbb{Z}\langle\langle X_1, \ldots, X_n \rangle\rangle / \sim_G$$

Independence — a rule for specifying joint expectations in terms of pure expectations

$\iff$ a construction of product states

$$\begin{tikzcd}
\mathcal{A}_i \arrow[r, hook] \arrow[d, "\varphi_i"'] & \mathop{*}_i \mathcal{A}_i \arrow[d] \arrow[dl, dotted, "\mathop{*}_i \varphi_i"] \\
\mathbb{C} & \bigotimes_i \mathcal{A}_i \arrow[l, dotted, "\bigotimes_i \varphi_i"]
\end{tikzcd}$$

$$(\varphi_1 \otimes \varphi_2)(X_1^n X_2^m) = \varphi_1(X_1^n)\varphi_2(X_2^m)$$

$$(\varphi_1 * \varphi_2)\Big(\big(X_1^{a_1} - \varphi_1(X_1^{a_1})\big)\cdots\big(X_2^{a_{2k}} - \varphi_2(X_2^{a_{2k}})\big)\Big) = 0$$

Example:

$\mathcal{H} = L^2([0,\infty), dx)$

$\mathcal{F}_0(\mathcal{H}) = \bigoplus_{k \geq 0} \mathcal{H}^{\otimes k}$ full free Fock space

$\mathcal{F}_1(\mathcal{H}) = \bigoplus_{k \geq 0} \mathcal{H}^{\otimes k} / \sim$ symmetric Fock space

$\varphi_i : B(\mathcal{H}) \to \mathbb{C} \quad : \quad T \longmapsto \langle \Omega, T\Omega \rangle$

$l_i(t) \in B(\mathcal{F}_i(\mathcal{H})) \quad : \quad \xi_1 \otimes \cdots \otimes \xi_k \longmapsto \chi_{[0,t]} \otimes \xi_1 \otimes \cdots \otimes \xi_k$

$g_i(t) = l_i(t) + l_i(t)^*$ under φ_i is

free / classical Brownian motion

Example:

$$\mathcal{H} = L^2([0,\infty), dx)$$

$$\mathcal{F}_0(\mathcal{H}) = \bigoplus_{k\geq 0} \mathcal{H}^{\otimes k} \quad \text{full free Fock space}$$

$$\mathcal{F}_1(\mathcal{H}) = \bigoplus_{k\geq 0} \mathcal{H}^{\otimes k}/\sim \quad \text{symmetric Fock space}$$

$$\varphi_i : B(\mathcal{H}) \to \mathbb{C} \ : \ T \longmapsto \langle \Omega, T\Omega \rangle$$

$$l_i(t) \in B(\mathcal{F}_i(\mathcal{H})) \ : \ \xi_1 \otimes \cdots \otimes \xi_k \longmapsto \chi_{[0,t]} \otimes \xi_1 \otimes \cdots \otimes \xi_k$$

$g_i(t) = l_i(t) + l_i(t)^*$ under φ_i is

free / classical Brownian motion

Example:

$$\mathcal{H} = L^2([0,\infty), dx)$$

$$\mathcal{F}_0(\mathcal{H}) = \bigoplus_{k\geq 0} \mathcal{H}^{\otimes k} \quad \text{full free Fock space}$$

$$\mathcal{F}_1(\mathcal{H}) = \bigoplus_{k\geq 0} \mathcal{H}^{\otimes k}/\sim \quad \text{symmetric Fock space}$$

$$\varphi_i : B(\mathcal{H}) \to \mathbb{C} \quad : \quad T \longmapsto \langle \Omega, T\Omega \rangle$$

$$l_i(t) \in B(\mathcal{F}_i(\mathcal{H})) \quad : \quad \xi_1 \otimes \cdots \otimes \xi_k \longmapsto \chi_{[0,t]} \otimes \xi_1 \otimes \cdots \otimes \xi_k$$

$g_i(t) = l_i(t) + l_i(t)^*$ under φ_i is

free / classical Brownian motion

Example:

$$\mathcal{H} = L^2([0,\infty), dx)$$

$$\mathcal{F}_0(\mathcal{H}) = \bigoplus_{k \geq 0} \mathcal{H}^{\otimes k} \qquad \text{full free Fock space}$$

$$\mathcal{F}_1(\mathcal{H}) = \bigoplus_{k \geq 0} \mathcal{H}^{\otimes k} / \sim \qquad \text{symmetric Fock space}$$

$$\varphi_i : B(\mathcal{H}) \to \mathbb{C} \quad : \quad T \longmapsto \langle \Omega, T\Omega \rangle$$

$$l_i(t) \in B(\mathcal{F}_i(\mathcal{H})) \ : \ \xi_1 \otimes \cdots \otimes \xi_k \longmapsto \chi_{[0,t]} \otimes \xi_1 \otimes \cdots \otimes \xi_k$$

$g_i(t) = l_i(t) + l_i(t)^*$ under φ_i is

free / classical Brownian motion

Independences

[Bożejko — Speicher 1991, ...]

q-deformed Fock space

$$\langle \xi_1 \otimes \cdots \otimes \xi_k, \eta_1 \otimes \cdots \otimes \eta_k \rangle_q = \sum_{\sigma \in S_k} q^{i(\sigma)} \langle \xi_1, \eta_{\sigma(1)} \rangle \cdots \langle \xi_k, \eta_{\sigma(k)} \rangle$$

$\ell_q(t) + \ell_q(t)^*$ q-deformed Brownian motion

Independences

[Bożejko — Speicher 1991, ...]

q-deformed Fock space

$$\langle \xi_1 \otimes \cdots \otimes \xi_k, \eta_1 \otimes \cdots \otimes \eta_k \rangle_q = \sum_{\sigma \in S_k} q^{i(\sigma)} \langle \xi_1, \eta_{\sigma(1)} \rangle \cdots \langle \xi_k, \eta_{\sigma(k)} \rangle$$

$l_q(t) + l_q(t)^*$ q-deformed Brownian motion

Independences

[Bożejko — Speicher 1991, ...]

q-deformed Fock space

$$\langle \xi_1 \otimes \cdots \otimes \xi_k, \eta_1 \otimes \cdots \otimes \eta_k \rangle_q$$

$$= \sum_{\sigma \in S_k} q^{i(\sigma)} \langle \xi_1, \eta_{\sigma(1)} \rangle \cdots \langle \xi_k, \eta_{\sigma(k)} \rangle$$

$\ell_q(t) + \ell_q(t)^*$ q-deformed Brownian motion

Are these central limit distributions

for a q-deformed free independence?

[Bożejko — Speicher 1991, ...]

q-deformed Fock space

$$\langle \xi_1 \otimes \cdots \otimes \xi_k, \eta_1 \otimes \cdots \otimes \eta_k \rangle_q = \sum_{\sigma \in S_k} q^{i(\sigma)} \langle \xi_1, \eta_{\sigma(1)} \rangle \cdots \langle \xi_k, \eta_{\sigma(k)} \rangle$$

$l_q(t) + l_q(t)^*$ q-deformed Brownian motion

Are these central limit distributions for a q-deformed free independence?

[Speicher; Muraki c. 2000s] No.

Are these central limit distributions for a q-deformed free independence?

[Speicher; Muraki c. 2000s] No.

There are 5 "reasonable" ways of defining a product state.

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r, hook] \arrow[d, "\varphi_1"'] & \mathcal{A}_1 * \mathcal{A}_2 \arrow[dl, dotted] \\
\mathbb{C} & \mathcal{A}_2 \arrow[l, "\varphi_2"] \arrow[u, hook]
\end{tikzcd}$$

(Only 3 of them are symmetric. Only $*$ and $\otimes$ are symmetric and unital.)

But on the other hand... [Młotkowski 2004]

$\Lambda = [n]$ $\quad G = (\Lambda, E)$ simple graph

$(\mathcal{A}_v, \varphi_v)_{v \in \Lambda}$ tracial $*$-algebras

$$(\mathcal{A}, \varphi) = \circledast_G (\mathcal{A}_v, \varphi_v)$$

- generated by (copies of) $\mathcal{A}_1, \ldots, \mathcal{A}_n$
- if $K \subseteq \Lambda$ is a clique, $(\mathcal{A}_v)_{v \in K}$ pairwise commute and are $\otimes$-independent
- if $\mathcal{K} \subseteq \Lambda$ is an anticlique, $(\mathcal{A}_v)_{v \in \mathcal{K}}$ are freely independent

The q-deformed Gaussians are the output of a "G-central limit theorem".

Independences

But on the other hand... [Młotkowski 2004]

$\Lambda = [n]$ $\qquad$ $G = (\Lambda, E)$ simple graph

$(\mathcal{A}_v, \varphi_v)_{v \in \Lambda}$ tracial $*$-algebras

$(\mathcal{A}, \varphi) = \underset{G}{\circledast} (\mathcal{A}_v, \varphi_v)$

- generated by (copies of) $\mathcal{A}_1, \ldots, \mathcal{A}_n$
- if $K \subseteq \Lambda$ is a clique, $(\mathcal{A}_v)_{v \in K}$ pairwise commute and are $\otimes$-independent
- if $\mathcal{K} \subseteq \Lambda$ is an anticlique, $(\mathcal{A}_v)_{v \in \mathcal{K}}$ are freely independent

The q-deformed Gaussians are the output of a "G-central limit theorem".

Independences

But on the other hand... [Młotkowski 2004]

$\Lambda = [n]$ $\quad$ $G = (\Lambda, E)$ simple graph

$(\mathcal{A}_v, \varphi_v)_{v \in \Lambda}$ tracial $*$-algebras

$(\mathcal{A}, \varphi) = \circledast_G (\mathcal{A}_v, \varphi_v)$

- generated by (copies of) $\mathcal{A}_1, \ldots, \mathcal{A}_n$
- if $K \subseteq \Lambda$ is a clique, $(\mathcal{A}_v)_{v \in K}$ pairwise commute and are $\otimes$-independent
- if $\mathcal{K} \subseteq \Lambda$ is an anticlique, $(\mathcal{A}_v)_{v \in \mathcal{K}}$ are freely independent

The q-deformed Gaussians are the output of a "G-central limit theorem".

Independences

But on the other hand... [Młotkowski 2004]

$\Lambda = [n]$ $G = (\Lambda, E)$ simple graph

$(\mathcal{A}_v, \varphi_v)_{v \in \Lambda}$ tracial $*$-algebras

$$(\mathcal{A}, \varphi) = \circledast_G (\mathcal{A}_v, \varphi_v)$$

- generated by (copies of) $\mathcal{A}_1, \ldots, \mathcal{A}_n$
- if $K \subseteq \Lambda$ is a clique, $(\mathcal{A}_v)_{v \in K}$ pairwise commute and are $\otimes$-independent
- if $\mathcal{K} \subseteq \Lambda$ is an anticlique, $(\mathcal{A}_v)_{v \in \mathcal{K}}$ are freely independent

The q-deformed Gaussians are the output of a "G-central limit theorem".

But on the other hand... [Młotkowski 2004]

$\Lambda = [n]$ $\quad G = (\Lambda, E)$ simple graph

$(\mathcal{A}_v, \varphi_v)_{v \in \Lambda}$ tracial $*$-algebras

$$(\mathcal{A}, \varphi) = \underset{G}{\circledast} (\mathcal{A}_v, \varphi_v)$$

- generated by (copies of) $\mathcal{A}_1, \dots, \mathcal{A}_n$
- if $K \subseteq \Lambda$ is a clique, $(\mathcal{A}_v)_{v \in K}$ pairwise commute and are $\otimes$-independent
- if $\mathcal{K} \subseteq \Lambda$ is an anticlique, $(\mathcal{A}_v)_{v \in \mathcal{K}}$ are freely independent

The q-deformed Gaussians are the output of a "G-central limit theorem".

Independences

But on the other hand... [Młotkowski 2004]

$\Lambda = [n]$ $\quad$ $G = (\Lambda, E)$ simple graph

$(\mathcal{A}_v, \varphi_v)_{v \in \Lambda}$ tracial $*$-algebras

$$(\mathcal{A}, \varphi) = \underset{G}{\circledast} (\mathcal{A}_v, \varphi_v)$$

- generated by (copies of) $\mathcal{A}_1, \ldots, \mathcal{A}_n$
- if $K \subseteq \Lambda$ is a clique, $(\mathcal{A}_v)_{v \in K}$ pairwise commute and are $\otimes$-independent
- if $\mathcal{K} \subseteq \Lambda$ is an anticlique, $(\mathcal{A}_v)_{v \in \mathcal{K}}$ are freely independent

The q-deformed Gaussians are the output of a "G-central limit theorem".

Independences

$\mathop{\circledast}_{v \in G} A_v$ faithfully represented on

$$\bigoplus_{\substack{[w] \in A^G \\ G\text{-reduced}}} \mathring{A}_{w_1} \otimes \cdots \otimes \mathring{A}_{w_k} \qquad \text{where} \quad \mathring{A}_w = \ker \varphi_w$$

$[w]$ is G-reduced if it has no representative with adjacent matching letters.

E.g. 

2342 ✓ 2312 ✗

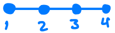
Main idea: if $A^G_v = \{ [w] \in A^G \mid w_1 = v \}$ then

$$\bigoplus_{\substack{[w] \in A^G \\ G\text{-reduced}}} \mathring{A}_{w_1} \otimes \cdots \otimes \mathring{A}_{w_k} \cong A_v \otimes \left(\bigoplus_{[w] \in A^G \setminus A^G_v} \mathring{A}_{w_1} \otimes \cdots \otimes \mathring{A}_{w_k} \right)$$

$\bigotimes_{v \in G} A_v$ faithfully represented on

$$\bigoplus_{\substack{[w] \in A^G \\ G\text{-reduced}}} \mathring{A}_{w_1} \otimes \cdots \otimes \mathring{A}_{w_k} \quad \text{where} \quad \mathring{A}_w = \ker \varphi_w$$

$[w]$ is G-reduced if it has no representative with adjacent matching letters.

E.g.  2342 ✓ 2312 ✗

Main idea: if $A^G_v = \{[w] \in A^G \mid w_1 = v\}$ then

$$\bigoplus_{\substack{[w] \in A^G \\ G\text{-reduced}}} \mathring{A}_{w_1} \otimes \cdots \otimes \mathring{A}_{w_k} \cong A_v \otimes \Big(\bigoplus_{[w] \in A^G \setminus A^G_v} \mathring{A}_{w_1} \otimes \cdots \otimes \mathring{A}_{w_k} \Big)$$

Independences

$\underset{v \in G}{\circledast} A_v$ faithfully represented on

$$\bigoplus_{\substack{[w] \in A^G \\ G\text{-reduced}}} \mathring{A}_{w_1} \otimes \cdots \otimes \mathring{A}_{w_k} \qquad \text{where} \quad \mathring{A}_w = \ker \varphi_w$$

$[w]$ is G-reduced if it has no representative with adjacent matching letters.

E.g. 

2342 ✓ 2312 ✗

Main idea: if $A^G_v = \{[w] \in A^G \mid w_1 = v\}$ then

$$\bigoplus_{\substack{[w] \in A^G \\ G\text{-reduced}}} \mathring{A}_{w_1} \otimes \cdots \otimes \mathring{A}_{w_k} \cong A_v \otimes \Big(\bigoplus_{[w] \in A^G \setminus A^G_v} \mathring{A}_{w_1} \otimes \cdots \otimes \mathring{A}_{w_k} \Big)$$

[Cébron — Oliveira Santos — Youssef]

G_ι sequence of graphs converging to a graphon $\mathfrak{W}$

$(x_{v,\iota})_{v \in V(G_\iota)}$ G_ι-independent and all identically distributed with mean 0, variance 1.

Then

$$\frac{1}{\sqrt{|V(G_\iota)|}} \sum_{v \in V(G_\iota)} x_v \xrightarrow{\text{dist}} x_{\mathfrak{W}} \text{ depending only on } \mathfrak{W}$$

Products of Groups

[Green 1990]

$G = (A, E)$ $\qquad (\Gamma_v)_{v\in A}$ groups

$$\bigstar_{v\in G} \Gamma_v = \mathop{*}_{v\in V(G)} \Gamma_v \Big/ \Big\langle [\Gamma_u, \Gamma_v] \,\Big|\, (u,v)\in E \Big\rangle$$

Eg. Right-angled Artin groups $\bigstar_{G} \mathbb{Z}$

[Caspers – Fima 2017]

Graph product defined for C^*, W^*-algebras

$$\bigstar_v L(\Gamma_v) = L\Big(\bigstar_v \Gamma_v\Big)$$

Preservation of Haagerup property, ...

Products of Groups

[Green 1990]

$G = (A, E)$ $\qquad (\Gamma_v)_{v \in A}$ groups

$$\mathop{\circledast}_{v \in G} \Gamma_v = \mathop{*}_{v \in V(G)} \Gamma_v \Big/ \left\langle [\Gamma_u, \Gamma_v] \,\middle|\, (u,v) \in E \right\rangle$$

Eg. Right-angled Artin groups $\mathop{\circledast}_{G} \mathbb{Z}$

[Caspers — Fima 2017]

Graph product defined for C^*, W^*-algebras

$$\mathop{\circledast}_{v} L(\Gamma_v) = L\left(\mathop{\circledast}_{v} \Gamma_v\right)$$

Preservation of Haagerup property, ...

Matrix models

x_1, x_2 random variables

$X_{i,N}$ $N \times N$ matrices $\qquad X_{i,N} \xrightarrow{\text{dist}} x_i$

$$\left(\frac{1}{N} \mathrm{Tr}(X_{i,N}^k) \longrightarrow \mathbb{E}[x_i^k] \right)$$

$U_{i,N}$ independent uniformly random $N \times N$ unitaries

$$Y_{i,N} = U_{i,N} X_{i,N} U_{i,N}^*$$

$$(Y_{1,N} \otimes 1_N, 1_N \otimes Y_{2,N}) \xrightarrow{\text{dist}} (x_1, x_2) \quad \text{classically independent}$$

$$(Y_{1,N}, Y_{2,N}) \xrightarrow{\text{dist}} (x_1, x_2) \quad \text{freely independent}$$

Matrix models

x_1, x_2 random variables

$X_{i,N}$ $N \times N$ matrices $\qquad X_{i,N} \xrightarrow{\text{dist}} x_i$

$\left(\frac{1}{N} \mathrm{Tr}(X_{i,N}^k) \longrightarrow \mathbb{E}[x_i^k] \right)$

$U_{i,N}$ independent uniformly random $N \times N$ unitaries

$Y_{i,N} = U_{i,N} X_{i,N} U_{i,N}^*$

$(Y_{1,N} \otimes 1_N, 1_N \otimes Y_{2,N}) \xrightarrow{\text{dist}} (x_1, x_2)$ classically independent

$(Y_{1,N}, Y_{2,N}) \xrightarrow{\text{dist}} (x_1, x_2)$ freely independent

Matrix models

x_1, x_2 random variables

$X_{i,N}$ $N \times N$ matrices

$X_{i,N} \xrightarrow{\text{dist}} x_i$

$\left(\frac{1}{N} \mathrm{Tr}(X_{i,N}^k) \longrightarrow \mathbb{E}[x_i^k] \right)$

$U_{i,N}$ independent uniformly random $N \times N$ unitaries

$Y_{i,N} = U_{i,N} X_{i,N} U_{i,N}^*$

$(Y_{1,N} \otimes 1_N, 1_N \otimes Y_{2,N}) \xrightarrow{\text{dist}} (x_1, x_2)$ classically independent

$(Y_{1,N}, Y_{2,N}) \xrightarrow{\text{dist}} (x_1, x_2)$ freely independent

Matrix models

x_1, x_2 random variables

$X_{i,N}$ $N \times N$ matrices $\qquad X_{i,N} \xrightarrow{\text{dist}} x_i$

$\left(\frac{1}{N} \mathrm{Tr}(X_{i,N}^k) \longrightarrow \mathbb{E}[x_i^k] \right)$

$U_{i,N}$ independent uniformly random $N \times N$ unitaries

$Y_{i,N} = U_{i,N} X_{i,N} U_{i,N}^*$

$(Y_{1,N} \otimes 1_N, 1_N \otimes Y_{2,N}) \xrightarrow{\text{dist}} (x_1, x_2)$ classically independent

$(Y_{1,N}, Y_{2,N}) \xrightarrow{\text{dist}} (x_1, x_2)$ freely independent

Matrix models

x_1, x_2 random variables

$X_{i,N}$ $N \times N$ matrices

$X_{i,N} \xrightarrow{\text{dist}} x_i$

$\left(\frac{1}{N} \mathrm{Tr}(X_{i,N}^k) \longrightarrow \mathbb{E}[x_i^k] \right)$

$U_{i,N}$ independent uniformly random $N \times N$ unitaries

$Y_{i,N} = U_{i,N} X_{i,N} U_{i,N}^*$

$(Y_{1,N} \otimes 1_N, 1_N \otimes Y_{2,N}) \xrightarrow{\text{dist}} (x_1, x_2)$ classically independent

$(Y_{1,N}, Y_{2,N}) \xrightarrow{\text{dist}} (x_1, x_2)$ freely independent

[C — Collins; C — de Santiago — Hayes — Jekel — Kunnawalkam Elayavalli; Morampudi - Laumann]

Using more tensor factors and unitaries uniform in well-chosen subgroups leads to G-independence.

This can be adapted to random permutation matrices (with care over the diagonal) which tells us graph products of type I factors with vanishing first L^2 Betti number are strongly 1-bounded (and in particular, not free group factors).

(c.f. [Davis — Okun])

Matrix models

[C — Collins; C — de Santiago — Hayes — Jekel — Kunnawalkam Elayavalli; Morampudi - Laumann]

Using more tensor factors and unitaries uniform in well-chosen subgroups leads to $\mathcal{G}$-independence.

This can be adapted to random permutation matrices (with care over the diagonal) which tells us graph products of type I factors with vanishing first L^2 Betti number are strongly 1-bounded (and in particular, not free group factors).

(c.f. [Davis — Okun])

Central summands

When can we have type I summands in $\underset{v\in G}{\circledast} \mathcal{A}_v$?

Subcases:

- If $G = K_n$, $\underset{v\in G}{\circledast} \mathcal{A}_v = \bigotimes_{v\in V(G)} \mathcal{A}_v$ so the type I summands are tensor products of those in the $\mathcal{A}_v$.

- [Dykema 1993; Ueda 2011]

  If $G = \overline{K}_n$, $\underset{v\in G}{\circledast} \mathcal{A}_v = \mathop{\ast}_{v\in V(G)} \mathcal{A}_v$.

  $\mathbb{C}$ occurs as direct summand with weight $\alpha \in (0,1]$ if each $\mathcal{A}_v$ has $\mathbb{C}$ as a direct summand with weight α_v, and $1-\alpha = (1-\alpha_1) + \cdots + (1-\alpha_n)$

  (Ueda characterizes types of all summands, but this margin is too s

Central summands

When can we have type I summands in $\bigotimes_{v \in G} \mathcal{A}_v$?

Subcases:

- If $G = K_n$, $\bigotimes_{v \in G} \mathcal{A}_v = \bigotimes_{v \in V(G)} \mathcal{A}_v$ so the type I summands are tensor products of those in the $\mathcal{A}_v$.

- [Dykema 1993; Ueda 2011]

  If $G = \overline{K_n}$, $\bigotimes_{v \in G} \mathcal{A}_v = \mathop{*}_{v \in V(G)} \mathcal{A}_v$.

  $\mathbb{C}$ occurs as direct summand with weight $\alpha \in (0, 1]$ if each $\mathcal{A}_v$ has $\mathbb{C}$ as a direct summand with weight α_v, and $1 - \alpha = (1 - \alpha_1) + \cdots + (1 - \alpha_n)$

  (Ueda characterizes types of all summands, but this margin is too s

Central summands

When can we have type I summands in $\circledast_{v \in G} \mathcal{A}_v$?

Subcases:

- If $G = K_n$, $\circledast_{v \in G} \mathcal{A}_v = \bigotimes_{v \in V(G)} \mathcal{A}_v$ so the type I summands are tensor products of those in the $\mathcal{A}_v$.

- [Dykema 1993; Ueda 2011]
  If $G = \overline{K}_n$, $\circledast_{v \in G} \mathcal{A}_v = \ast_{v \in V(G)} \mathcal{A}_v$.
  $\mathbb{C}$ occurs as direct summand with weight $\alpha \in (0,1]$
  if each $\mathcal{A}_v$ has $\mathbb{C}$ as a direct summand
  with weight α_v, and $1-\alpha = (1-\alpha_1) + \cdots + (1-\alpha_n)$

  (Ueda characterizes types of all summands, but this margin is too s

Central summands

Aside:

x_1, x_2 projections of trace $\frac{1}{2} \leq \alpha_1 \leq \alpha_2 < 1$

Then:

$$W^*(x_1) * W^*(x_2) \cong \begin{cases} \text{something diffuse} & \text{if } \frac{1}{2} = \alpha_1 = \alpha_2 \\ \mathbb{C} \oplus \text{something diffuse} & \text{if } \frac{1}{2} < \alpha_1 = \alpha_2 \\ \mathbb{C} \oplus \mathbb{C} \oplus \text{something diffuse} & \text{if } \alpha_1 < \alpha_2 \end{cases}$$

$\Downarrow$

$x_1 + x_2$ has corresponding atoms in its spectral measure

$W^*(x_1) \otimes W^*(x_2) \cong \mathbb{C}^2 \otimes \mathbb{C}^2$ has 4 summands

$\Downarrow$

$x_1 + x_2$ has 3 atoms

Central summands

Simpler question: $p_v \in \mathcal{A}_v$ for each $v \in A$.

When is $p = \bigwedge_{v \in A} p_v \neq 0$ in $\mathop{\circledast}_{v \in G} \mathcal{A}_v$?

For a graph H with $V(H) = \{1, \ldots, h\}$ let

$$\mathfrak{K}_H(X_1, \ldots, X_h) = \sum_{\substack{K \subseteq H \\ \text{clique}}} \prod_{v \in K} -X_v \in \mathbb{C}[[X_1, \ldots, X_h]]$$

Theorem [C – Jekel]

$p \neq 0 \iff \forall A' \subseteq A$, writing $G' \leq G$ for the corresponding induced subgraph,

$$\mathfrak{K}_{G'}\left((1 - \varphi(p_v))_{v \in A'} \right) > 0 .$$

In this case, this is equal to $\varphi(p)$.

Central summands

Simpler question: $p_v \in \mathcal{A}_v$ for each $v \in A$.

When is $p = \bigwedge_{v \in A} p_v \neq 0$ in $\mathop{\circledast}_{v \in G} \mathcal{A}_v$?

For a graph H with $V(H) = \{1, \dots, h\}$ let

$$\mathfrak{K}_H(X_1, \dots, X_h) = \sum_{\substack{K \subseteq H \\ \text{clique}}} \prod_{v \in K} -X_v \in \mathbb{C}[[X_1, \dots, X_h]]$$

Theorem [C – Jekel]

$p \neq 0 \iff \forall A' \subseteq A$, writing $G' \leq G$ for the corresponding induced subgraph,

$$\mathfrak{K}_{G'}\Big(\big(1 - \varphi(p_v)\big)_{v \in A'} \Big) > 0 .$$

In this case, this is equal to $\varphi(p)$.

Simpler question: $p_v \in \mathcal{A}_v$ for each $v \in A$.

When is $p = \bigwedge_{v \in A} p_v \neq 0$ in $\underset{v \in G}{\circledast} \mathcal{A}_v$?

For a graph H with $V(H) = \{1, \ldots, h\}$ let

$$\mathfrak{K}_H(X_1, \ldots, X_h) = \sum_{\substack{K \subseteq H \\ \text{clique}}} \prod_{v \in K} -X_v \in \mathbb{C}[[X_1, \ldots, X_h]]$$

\mathfrak{K}

Theorem [C – Jekel]

$p \neq 0 \iff \forall A' \subseteq A$, writing $G' \leq G$ for the corresponding induced subgraph,

$$\mathfrak{K}_{G'}\left((1 - \varphi(p_v))_{v \in A'} \right) > 0 .$$

In this case, this is equal to $\varphi(p)$.

Central summands

Simpler question: $p_v \in \mathcal{A}_v$ for each $v \in A$.

When is $p = \bigwedge_{v \in A} p_v \neq 0$ in $\circledast_{v \in G} \mathcal{A}_v$?

For a graph H with $V(H) = \{1, \ldots, n\}$ let

$$\mathfrak{K}_H(X_1, \ldots, X_n) = \sum_{\substack{K \subseteq H \\ \text{clique}}} \prod_{v \in K} -X_v \in \mathbb{C}[[X_1, \ldots, X_n]]$$

Theorem [C – Jekel]

$p \neq 0 \iff \forall A' \subseteq A$, writing $G' \leq G$ for the corresponding induced subgraph,

$$\mathfrak{K}_{G'}\left((1 - \varphi(p_v))_{v \in A'} \right) > 0 .$$

In this case, this is equal to $\varphi(p)$.

Combinatorics of words

[Cartier — Foata 1969, Paraphrased]

Let $G = (A, E)$ be a (simple) graph

$$A^G = A^* / \{uv \sim vu \mid (u,v) \in E\}$$

$12321 \sim 2^2131 \sim 132^21$

$1234 \sim 1324$

$$\sum_{[w]\in A^G} X_w = \Bigg(\overbrace{\sum_{\substack{K \subseteq A \\ \text{clique}}} \prod_{v \in K} (-X_v)}^{\kappa_G(X_1,\dots,X_n)} \Bigg)^{-1} \in \mathbb{Z}[[X_1,\dots,X_n]]$$

or

$$\mathbb{Z}\langle\langle X_1,\dots,X_n\rangle\rangle / \sim_G$$

Free etymology

$$\mathcal{H} = \overline{\bigoplus_{\substack{[w]\in A^G\\ G\text{-reduced}}} \mathring{A}_{w_1}\otimes\cdots\otimes\mathring{A}_{w_k}} \qquad \text{where } \mathring{A}_w = \ker\varphi_w$$

Assume $A_v = \mathbb{C}p_v \oplus \mathbb{C}(1-p_v)$; write $\mathring{p}_v = \dfrac{p_v - \varphi(p_v)}{\|p_v - \varphi(p_v)\|}$ so $\mathring{A}_v = \mathbb{C}\mathring{p}_v$

Suppose $\xi = \sum\limits_{\substack{[w]\in A^G\\ G\text{-reduced}}} \xi([w])\, \mathring{p}_{w_1}\otimes\cdots\otimes\mathring{p}_{w_k} \in p_v\mathcal{H}$

$$p_v(1) = p_v - \varphi(p_v) + \varphi(p_v) = \|p_v - \varphi(p_v)\|\,\mathring{p}_v + \varphi(p_v)1$$

$$p_v(\mathring{p}_v) = (1-\varphi(p_v))\mathring{p}_v + \|p_v - \varphi(p_v)\|\,1$$

$$\Rightarrow \xi([v]) = \sqrt{\frac{1-\varphi(p_v)}{\varphi(p_v)}}\,\xi([\,])$$

Similarly if $[w]\in A^G\setminus A^G_v$ $\quad \xi([vw]) = \sqrt{\dfrac{1-\varphi(p_v)}{\varphi(p_v)}}\,\xi([w])$

Free etymology

$$\mathcal{H} = \overline{\bigoplus_{\substack{[w]\in A^G \\ G\text{-reduced}}} \mathring{A}_{w_1} \otimes \cdots \otimes \mathring{A}_{w_k}} \quad \text{where} \quad \mathring{A}_w = \ker \varphi_w$$

Assume $A_v = \mathbb{C}p_v \oplus \mathbb{C}(1-p_v)$; write $\mathring{p}_v = \dfrac{p_v - \varphi(p_v)}{\|p_v - \varphi(p_v)\|}$ so $\mathring{A}_v = \mathbb{C}\mathring{p}_v$

Suppose $\xi = \sum_{\substack{[w]\in A^G \\ G\text{-reduced}}} \xi([w])\, \mathring{p}_{w_1} \otimes \cdots \otimes \mathring{p}_{w_k} \in p_v \mathcal{H}$

$$p_v(1) = p_v - \varphi(p_v) + \varphi(p_v) = \|p_v - \varphi(p_v)\| \mathring{p}_v + \varphi(p_v) 1$$

$$p_v(\mathring{p}_v) = (1 - \varphi(p_v))\mathring{p}_v + \|p_v - \varphi(p_v)\| 1$$

$$\Rightarrow \xi([v]) = \sqrt{\frac{1-\varphi(p_v)}{\varphi(p_v)}}\, \xi([\,])$$

Similarly if $[w] \in A^G \setminus A^G_v$ $\quad \xi([vw]) = \sqrt{\dfrac{1-\varphi(p_v)}{\varphi(p_v)}}\, \xi([w])$

Free etymology

$$\mathcal{H} = \overline{\bigoplus_{\substack{[w]\in A^G \\ G\text{-reduced}}} \mathring{A}_{w_1} \otimes \cdots \otimes \mathring{A}_{w_k}} \quad \text{where} \quad \mathring{A}_w = \ker \varphi_w$$

Assume $A_v = \mathbb{C}p_v \oplus \mathbb{C}(1-p_v)$; write $\mathring{p}_v = \dfrac{p_v - \varphi(p_v)}{\|p_v - \varphi(p_v)\|}$ so $\mathring{A}_v = \mathbb{C}\mathring{p}_v$

Suppose $\xi = \sum_{\substack{[w]\in A^G \\ G\text{-reduced}}} \xi([w])\, \mathring{p}_{w_1} \otimes \cdots \otimes \mathring{p}_{w_k} \in p_v \mathcal{H}$

$$p_v(1) = p_v - \varphi(p_v) + \varphi(p_v) = \|p_v - \varphi(p_v)\|\, \mathring{p}_v + \varphi(p_v) 1$$

$$p_v(\mathring{p}_v) = (1 - \varphi(p_v))\mathring{p}_v + \|p_v - \varphi(p_v)\| 1$$

$$\Rightarrow \xi([v]) = \sqrt{\frac{1-\varphi(p_v)}{\varphi(p_v)}}\, \xi([\,])$$

Similarly if $[w] \in A^G \setminus A^G_v$ $\quad \xi([vw]) = \sqrt{\dfrac{1-\varphi(p_v)}{\varphi(p_v)}}\, \xi([w])$

Free etymology

$$\xi = \sum_{\substack{[w]\in A^G \\ G\text{-reduced}}} \xi([w])\, \mathring{p}_{w_1} \otimes \cdots \otimes \mathring{p}_{w_k} \in p_v \mathcal{H} \Rightarrow \xi([vw]) = \sqrt{\frac{1-\varphi(p_v)}{\varphi(p_v)}}\, \xi([w])$$

If we normalize $\xi([\,]) = 1$ then

$$\xi([w]) = \prod_i \sqrt{\frac{1-\varphi(p_{w_i})}{\varphi(p_{w_i})}}$$

We need

$$\|\xi\|^2 = \sum_{\substack{[w]\in A^G \\ G\text{-reduced}}} |\xi([w])|^2 = \sum_{\substack{[w]\in A^G \\ G\text{-reduced}}} \prod_i \frac{1-\varphi(p_{w_i})}{\varphi(p_{w_i})} < \infty$$

Free etymology

$$\xi = \sum_{\substack{[w]\in A^G \\ G\text{-reduced}}} \xi([w])\, \mathring{p}_{w_1} \otimes \cdots \otimes \mathring{p}_{w_k} \in p_v \mathcal{H} \implies \xi([vw]) = \sqrt{\frac{1-\varphi(p_v)}{\varphi(p_v)}}\, \xi([w])$$

If we normalize $\xi([\,]) = 1$ then

$$\xi([w]) = \prod_i \sqrt{\frac{1-\varphi(p_{w_i})}{\varphi(p_{w_i})}}$$

We need

$$\|\xi\|^2 = \sum_{\substack{[w]\in A^G \\ G\text{-reduced}}} |\xi([w])|^2 = \sum_{\substack{[w]\in A^G \\ G\text{-reduced}}} \prod_i \frac{1-\varphi(p_{w_i})}{\varphi(p_{w_i})} < \infty$$

Combinatorics of words

$$F(X_1, \ldots, X_n) = \sum_{[w] \in A^G} X_w \qquad F_r(X_1, \ldots, X_n) = \sum_{\substack{[w] \in A^G \\ G\text{-reduced}}} X_w$$

$$\Rightarrow \quad F(X_1, \ldots, X_n) = F_r\left(\frac{X_1}{1-X_1}, \ldots, \frac{X_n}{1-X_n}\right)$$

Free etymology

$$\|\xi\|^2 = \sum_{\substack{[w]\in A^G \\ G\text{-reduced}}} |\xi([w])|^2 = \sum_{\substack{[w]\in A^G \\ G\text{-reduced}}} \prod_i \frac{1-\varphi(p_{w_i})}{\varphi(p_{w_i})}$$

$$= \sum_{\substack{[w]\in A^G \\ G\text{-reduced}}} \prod_i \frac{1-\varphi(p_{w_i})}{1-(1-\varphi(p_{w_i}))}$$

$$= F_r\left(\frac{1-\varphi(p_1)}{1-(1-\varphi(p_1))}, \ldots, \frac{1-\varphi(p_n)}{1-(1-\varphi(p_n))}\right)$$

$$= F\left(1-\varphi(p_1), \ldots, 1-\varphi(p_n)\right)$$

$$\underset{(*)}{=} \mathfrak{K}_G\left(1-\varphi(p_1), \ldots, 1-\varphi(p_n)\right)^{-1}$$

Central summands

Theorem [C – Jekel]

Let $G = (A, E)$ be a finite simple graph,

$(\mathcal{A}_v, \varphi_v)_{v \in A}$ (faithful normal) statial vNas

The type I summands of $(\mathcal{A}, \varphi) = \mathop{\circledast}\limits_{v \in G} (\mathcal{A}_v, \varphi_v)$ can be explicitly described based on polynomial conditions $\mathfrak{K}_{G'} > 0$ at points depending on the summands of $\mathcal{A}_v$ and the restriction of φ_v to those summands.