$$\frac{F(t)-F(x_0)}{t-x_0} = \frac{1}{t-x_0}\int_{x_0}^{t} f(x)\,dx$$

$$= \underbrace{\frac{1}{t-x_0}\int_{x_0}^{t} f(x)-f(x_0)\,dx}_{} + \underbrace{\frac{1}{t-x_0}\int_{x_0}^{t} f(x_0)\,dx}_{}$$

$$= \frac{t-x_0}{t-x_0} f(x_0) = f(x_0)$$

Let $\varepsilon > 0$, choose δ s.t. $|f(x)-f(x_0)| < \varepsilon$ if $|x-x_0| < \delta$

then $|-| \le \frac{1}{|t-x_0|}\left|\int_{x_0}^{t} \varepsilon\,dx\right| = \varepsilon$.

Theorem (FTC):

Sps $f:[a,b]\to\mathbb{R}$ integrable & $\exists\ F:[a,b]\to\mathbb{R}$ cts, diffable with

$F' = f$ on (a,b).

Then $F(b)-F(a) = \int_a^b f(t)\,dt$

Pick a partition, use MVT on F,

$$F'(t_i) = f(t_i) \in [m_i, M_i]$$

Defn: Ptws conv

Q: $\lim_{n\to\infty}\lim_{t\to x} f_n(t) \overset{?}{=} \lim_{t\to x}\lim_{n\to\infty} f_n(t)$

No! E.g. $\left(\begin{array}{c}[0,1]\to\mathbb{R}\\ t\mapsto t^n\end{array}\right)_n$.

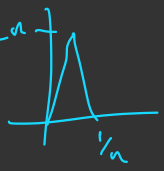
Q: $\lim_{n\to\infty} \int_a^b f_n(t)\,dt = \int_a^b \lim_{n\to\infty} f_n(t)\,dt$?

No! E.g. $(q_n)_n$ a sequence taking all values in $\mathbb{Q}$, $f_n : [0,1] \to \mathbb{R}$

$$t \longmapsto \begin{cases} 1 & \text{if } t = q_k \text{ for some } k \le n \\ 0 & \text{else} \end{cases}$$

What if we insist $\lim f_n$ is integrable?

No!



Many more bad eks.

Defn: unif conv

Thm: cts $\xrightarrow{u}$ cts

$\varepsilon/3$

Prop: u conv $\Rightarrow$ ptws conv

Thm: $f_n : S \to M$ unif Cauchy + M complete

$\Rightarrow$ f_n unif conv.

Def^n: $f: S \to \mathbb{R}$ bdd $\quad \|f\|_\infty$ $\quad \infty$ else

Thm: $f_n \xrightarrow{u} f$ iff $\|f_n - f\|_\infty \to 0$.

Thm: f_n integrable, u conv to f
$\Rightarrow f$ integrable and $\int_a^b f = \lim \int_a^b f_n$.

Don't assume $\int f_n$ converges!

Pf: $f_n - \varepsilon \le f \le f_n + \varepsilon$

$$\Rightarrow \int (f_n - \varepsilon) \le L(f) \le U(f) \le \int (f_n + \varepsilon)$$

$$\Rightarrow 0 \le U(f) - L(f) \le 2\int \varepsilon \to 0$$

$$0 \le \left| \int f - \int f_n \right| \le \int |f - f_n| < \int \varepsilon \to 0.$$

Thm (Stone's Theorem):
Sps $f:[a,b] \to \mathbb{R}$ cts. Then $\exists$ p_n poly w/ $p_n \xrightarrow{u} f$ on $[a,b]$.

Pf: WLOG $[a,b] = [0,1]$. $\quad (f \mapsto f(a + t(b-a)))$

WLOG $f(0) = 0 = f(1)$ $\quad (f \mapsto f - f(0) + t(f(0) - f(1)))$

Extend f to $\mathbb{R}$ by setting $f(t) = 0$ for $t \notin [0,1]$.

$Q_n(t) = c_n(1-x^2)^n$ where c_n is so that $\int_{-1}^{1} Q_n(t)\,dt = 1$

$$\frac{1}{c_n} = \int_{-1}^{1} (1-x^2)^n dt = 2\int_0^1 (1-t^2)^n dt \ge 2\int_0^{1/\sqrt{n}} (1-t^2)^n dt \ge 2\int_0^{1/\sqrt{n}} (1-nt^2)\,dt$$

Claim: $(1-nt^2) \le (1-t^2)^n$ on $[0, 1/\sqrt{n}]$

Set $d(t) = (1-t^2)^n - (1-nt^2)$. $d(0) = 0$, $d'(t) = -2nt(1-t^2)^{n-1} + 2nt > 0$

$$\frac{1}{c_n} \ge 2\int_0^{1/\sqrt{n}} (1-nt^2)\,dt = \frac{2}{\sqrt{n}} - \frac{2n}{3\sqrt{n^3}} = \frac{2}{\sqrt{n}}\left(1 - \frac{1}{3}\right) > \frac{1}{\sqrt{n}}$$

Hence $c_n < \sqrt{n}$

Now if $\delta > 0$ and $\delta < |x| < 1$,

$$Q_n(x) = c_n(1-x^2)^n < \sqrt{n}(1-\delta^2)^n$$

Set $P_n(x) = \int_{-1}^{1} f(x+t)\, Q_n(t)\, dt = \int_{-x}^{1-x} f(x+t) Q_n(t)\, dt = \int_0^1 f(t)\, Q_n(t-x)\, dt$

This last is a polynomial.

Now f is bounded and ucts on $[-50, 800]$. Let $\varepsilon > 0$ and choose $1 > \delta > 0$ so that $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon/2$.

Let $x \in [0,1]$.

$$|P_n(x) - f(x)| = \left| \int_{-1}^{1} f(x+t) Q_n(t)\, dt - f(x) \int_{-1}^{1} Q_n(t)\, dt \right|$$

$$\leq \int_{-1}^{1} |f(x+t) - f(x)|\, Q_n(t)\, dt \qquad (\text{since } Q_n(t) = |Q_n(t)|)$$

$$\leq \int_{-1}^{-\delta} 2\|f\|_\infty Q_n(t)\, dt + \int_{-\delta}^{\delta} \frac{\varepsilon}{2} Q_n(t)\, dt + \int_{\delta}^{1} 2\|f\|_\infty Q_n(t)\, dt$$

$$< 4\|f\|_\infty \sqrt{n}\,(1-\delta^2)^n + \frac{\varepsilon}{2} \int_{-1}^{1} Q_n(t)\, dt$$

Suffices to show $\lim_{n\to\infty} \sqrt{n}\,(1-\delta^2)^n = 0$

Let $\rho > 0$ be so that $(1-\delta)(1+\rho) = 1$.

Then $(1+\rho)^n = 1 + n\rho + \binom{n}{2}\rho^2 + \cdots \geq n\rho$

So $\sqrt{n}\,(1-\delta^2)^n = \frac{\sqrt{n}}{(1+\rho)^n} \leq \frac{\sqrt{n}}{n\rho} = \frac{1}{\sqrt{n}\rho} \longrightarrow 0$.

Taking n large enough to make $\sqrt{n}\,(1-\delta^2)^n < \frac{\varepsilon}{8\|f\|_\infty}$

we have $|P_n(x) - f(x)| < \varepsilon$. ∎

Corollary:

For any $a > 0$ there is a sequence of polynomials converging unif to $x \mapsto |x|$ on $[-a, a]$.